The Digits of 6174 at Every Length: Two Liftings of a Universal Kaprekar Fixed Point

Clay Elmore

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Abstract

The Kaprekar routine sends every four-digit number that is not a repdigit to 6174. We ask whether, for each length d, an arrangement of these digits, extended with zeros or by repetition, is a universal fixed point of one generalized Kaprekar rule, uniform in d. A finite-state criterion reduces universality to fixed-point uniqueness and acyclicity. An exhaustive census at $d \leq 6$ finds 0, 4, 33, and 506 universal full-variable fixed points; of the thirty-three at $d = 5$, only 60714 is again a universal full-variable fixed point at $d = 6$. For zeros, explicit ladder rules make padded 60714 universal at $d = 5, 6$ and, at every larger d, universal off an escape class, the inputs whose orbits reach 0, which grows overall with d (to 12% of multisets at $d = 24$). For repetition, folding preserves the fixed-point equation at every multiplicity and, by complete enumeration, often preserves universality: folds of the 1746 rule are universal at multiplicities 2, 4, and 5. For one dyadic tower we prove two conditions sufficient for universality and establish both at its first two levels; the general case is open. As one explanation of why the two directions differ, we prove that Kaprekar's four-digit theorem has no polynomial Lyapunov certificate below degree seven.

1 Introduction

Kaprekar observed in 1949 that the map which sorts the digits of a four-digit integer into descending and ascending order and subtracts sends every input with at least two distinct digits to 6174, in at most seven steps [Kaprekar 1949, Kaprekar 1955]. The three-digit analogue converges to 495 [Trigg 1974]; the five-digit analogue has no fixed point that attracts every input, every orbit falling into one of three cycles [Prichett 1981]. For more than seventy-five years these facts have been verified the same way they were discovered, by finite enumeration.

The classical routine is one member of a large family. Fix a length d, pad integers to d digits, and let any ordered pair of distinct permutations rearrange the sorted digit string before the subtraction; this yields $d!\,(d! - 1)$ ordered pairs at each length, of which Kaprekar's is one. A pair and its swap induce the same map, so there are at most $d!\,(d! - 1)/2$ distinct maps. For $d \geq 4$ there are fewer still, because two pairs that differ only by exchanging letters which each rearrangement puts in the same place also coincide. The distinct maps number 15, 240, 6,040 at $d = 3, 4, 5$, against 30, 552, 14,280 ordered pairs (Computed). The family is not new: Peterson and Pulapaka define these routines, by a pair of permutations of the digit positions, count them, and pose the classification of the four-digit ones as an open problem [Peterson 2008]. This paper asks whether the digits of 6174 keep their role across this family at every length: *for each d, is there an arrangement of those digits (extended either with zeros or by repetition) and a single rule, uniform in d, for which that*

arrangement is a universal fixed point, reached by every admissible input? The question has two directions, set by the two ways of extending the digits, and they behave very differently.

The generalized family has a growing literature, most of it fixing the length or the base and classifying the dynamics that result. Nuez studies parametric and algebraic families of Kaprekar-type maps and their symmetries [Nuez 2021, Nuez 2021b, Nuez 2022]; Kay and Downes-Ward determine the fixed points and cycles of the base-b transformation in odd and even bases [Kay 2022, Kay 2024]; Prichett's determination of the decadic constants, with its five-digit precursors, settles the classical fixed lengths [Prichett 1981]; Devlin and Zeng bound the four-digit reaching time [Devlin 2021]; Peterson and Pulapaka introduce the two-permutation family and survey the classical routine across bases [Peterson 2008]; and Dahl analyzes its basins and attractor entropy [Dahl 2026]. This paper takes a different approach: it fixes the *fixed point*, the digits of 6174, and varies the length, asking for a single rule, uniform across lengths, that keeps that fixed point universal. The object of study is the cross-dimensional lifting, not the classification at any one length.

Take the zeros first. Universality (every admissible input reaching the fixed point) is rare, and universality that survives a change of length is rarer. An exhaustive census at $d \leq 6$ (Section 3) finds four universal full-variable fixed points at $d = 4$, thirty-three at $d = 5$, five hundred six at $d = 6$; of the thirty-three, exactly one (60714, the digits of 6174 with a zero) is universal again at $d = 6$. That survivor extends to every length (Section 4) in the following sense. An explicit rule at each $d \geq 5$, the rungs linked by a coefficient-preserving ladder, makes 60714 padded with zeros universal at $d = 5, 6$. At every larger length it is universal off an escape class (Definition 4.6): the admissible inputs whose orbits reach 0, small at short lengths and growing overall with d, though not at every step. Its one-step part, the block-aligned multisets, has closed-form size $\binom{9+k}{k} - 10$, with k the number of coefficient blocks. The proof uses an absorbing set created by the padding zeros. It answers the zero direction for 60714 up to the escape class, which reaches 12% of multisets at $d = 24$ and extends to admissible inputs the collapse that the classical map confines to repdigits. We isolate this mechanism because the repetition direction lacks a uniform version of it.

Now the repetition. The same digits can instead be taken with multiplicity m (the multiset $\{1, 4, 6, 7\}^m$, a $4m$-digit problem). Universal fixed points of $\{1, 4, 6, 7\}^m$ exist at every $m \leq 6$, known by enumeration and extended here, including what is, to the author's knowledge, the first complete-basin verification at $d = 20$; they exist at $m = 7$ as well, by complete enumeration [N]. Whether they exist at every m is the open conjecture. Section 5 introduces a folding operation that makes the algebraic half of the question a theorem at all m, and Section 6 reports the central computational finding of this paper: folding preserves universality often (every measured doubling step has survival fraction 100% or 46%), and the folds of one four-digit rule give fully verified universal fixed points at multiplicities $1, 2, 4, 5$, with a single twelve-digit rule covering 3 and 6. Section 6 also finds what the surviving folds share. On a sign-coherent core that every orbit reaches within a few steps, a fold reduces to a finite pair system that funnels to its fixed point. This is the repetition direction's first absorbing mechanism, the analogue of the zeros in Section 4; its entry half is proven at the first two tower levels. The pattern of *which* folds survive is nonetheless irregular, and Section 7 measures what turning this abundance into a proof for all m would cost: the base case, Kaprekar's own theorem, carries no polynomial Lyapunov certificate below degree seven (exact Farkas certificates through degree six; an explicit degree-seven witness above). That is the cost of lacking a uniform absorbing mechanism, and it is the explanation we propose for why the two directions differ: zero-padding supplies an absorbing set at no cost, while repetition would need its base case to do the same, and the base case cannot do it cheaply. Closing the gap, for the one

tower where the coherent mechanism is already in hand, is the concrete open problem of Section 7.

Throughout, claims carry one of four labels. **Proven** means a mathematical proof appears here. **Computed** means a finite exhaustive enumeration, machine-verified; unless marked otherwise, every such enumeration was re-run independently for this paper (Section 9 details the few exceptions, marked [S], which are taken from the project's computational log, and also defines the tag [N]). **Evidence** means a claim that passes every feasible test but whose state space exceeds exhaustive reach. **Conjectured** means stated without proof or complete verification.

2 Rules, coefficients, and the universality criterion

Fix $d \geq 2$. For an integer n with at most d digits, write $\mathrm{sort}_{\downarrow}(n) = (s_0, \ldots, s_{d-1})$ for its digits padded to length d and sorted descending. For a permutation $\tau \in S_d$ let $\tau \cdot s$ denote the integer $\sum_k 10^{d-1-k} s_{\tau(k)}$.

Definition 2.1. *For an ordered pair $(\pi, \sigma) \in S_d \times S_d$ with $\pi \neq \sigma$, the rule $K_{\pi,\sigma}$ acts by*

$$K_{\pi,\sigma}(n) \;=\; \big|\, \pi \cdot \mathrm{sort}_{\downarrow}(n) \;-\; \sigma \cdot \mathrm{sort}_{\downarrow}(n) \,\big|.$$

The classical Kaprekar map is the pair (identity, reversal).

A rule is conveniently displayed by labeling the sorted positions $a = s_0,\ b = s_1,\ c = s_2, \ldots$ and writing the two rearrangements as letter-strings. The classical map at $d = 4$ is `abcd − dcba`: on the digits of 6174, sorted to $(a, b, c, d) = (7, 6, 4, 1)$, it computes $7641 - 1467 = 6174$. Rules quoted in the appendices use this notation.

Since the rule reads n only through its sorted digits, orbits live on digit multisets, and all state counts below are multiset counts. Expanding the difference,

$$K_{\pi,\sigma}(n) = \Big|\sum_{i=0}^{d-1} c_i\, s_i\,\Big|, \qquad c_i = 10^{\,d-1-\pi^{-1}(i)} - 10^{\,d-1-\sigma^{-1}(i)}, \qquad \sum_i c_i = 0.$$

The coefficient vector c determines the dynamics completely and is how rules are specified below; note that c and $-c$ induce the same map, so rules come in sign pairs with identical dynamics.

Lemma 2.2. *(Proven.) (a) $K_{\pi,\sigma}(n) \equiv 0 \pmod 9$ for every rule and every input. (b) The rules with $c_i \neq 0$ for all i (the* full-variable *rules) number $d!\, D_d$, where D_d is the number of derangements of d letters:* 12, 216, 5,280, 190,800 *at $d = 3, 4, 5, 6$.*

Proof. (a) Each c_i is a difference of two powers of 10, and $10^a \equiv 1 \pmod 9$. (b) $c_i = 0$ exactly when $\pi^{-1}(i) = \sigma^{-1}(i)$, so c has no zero entry exactly when $\sigma^{-1}\pi$ is fixed-point-free; for each of the $d!$ choices of π there are D_d such σ. ■

Two integer invariants grade a rule against a fixed point F with sorted digits $(f_0, \ldots, f_{d-1})$. The *rank* $\mathrm{sv}(K) = \#\{i : c_i \neq 0\}$ counts the sorted positions the rule reads; the *effective rank* $\mathrm{sv}_F(K) = \#\{i : c_i \neq 0 \text{ and } f_i \neq 0\}$ counts those that actually carry weight at F. A rule can be full-variable while $\mathrm{sv}_F < d$; the gap is invisible in the equation $K(F) = F$ and decisive for everything else (Section 3).

Definition 2.3. *F is* universal *for K at length d if $K(F) = F$ and every admissible input reaches F under iteration of K. Repdigits are always inadmissible (every rule sends them to 0). In the small-length censuses of Section 3 the ninety near-repdigits (one digit occurring $d-1$ times) are also*

excluded, following the classical convention; in the duplication chain of Sections 5–6 only repdigits are excluded. Each count below states its convention.

Everything in the paper passes through one elementary fact.

Lemma 2.4 (universality criterion). *(Proven.) Let T be the map $X \mapsto \mathrm{sort}_{\downarrow}K(X)$ on the set of* all *multisets of d digits, a finite set on which T is a genuine self-map, every repdigit (including 0) mapping to 0 and 0 fixed. Admissibility restricts only the starting states; orbits may pass through inadmissible multisets. Let $F \neq 0$ be a fixed multiset of T (the fixed integer is $|\sum_i c_i f_i|$, whose digit multiset is F). Then F is universal (reached by the orbit of every admissible input) if and only if no admissible orbit reaches a fixed multiset other than F (in particular none reaches 0) and no admissible orbit enters a periodic orbit of period ≥ 2.*

Proof. T is a self-map of a finite set, so every orbit is eventually periodic: it ends in a fixed multiset or a period-≥ 2 cycle, and that terminal set traps it. If F is universal, an admissible orbit reaching a different fixed multiset or entering a cycle would be trapped away from F, a contradiction; so neither occurs. Conversely, each admissible orbit ends in some terminal set, which by hypothesis can only be $\{F\}$. ■

Defining T on the full multiset space is not a formality: orbits do leave the admissible set in passing. Under the strict convention an admissible input can map to a near-repdigit (the ladder rule of Section 4 sends $\{9,9,9,9,9,8,7\}$ to 900000 at $d = 7$, the excursions Appendix E.4 tracks), and some rules send fully generic inputs to nonzero repdigits, which then collapse to 0: at $d = 3$ the multiset $\{9,6,2\}$ admits $962 - 629 = 333$. This cannot happen at $d = 4$ or $d = 5$, since every image is a multiple of 9, which forces the repdigit $10^d - 1$, and that exceeds the largest possible difference; it happens again at $d = 6$: $696329 - 362996 = 333333$ (Computed). The criterion therefore evaluates reachability in the full space. It makes universality decidable by finite search and splits its failure into three modes (escape to 0, a competing fixed point, or a cycle), all of which recur below; Section 5 exhibits a rule with two fixed points and cycles (the $m = 3$ interleaved rule), and the escape mode, empty in the $d \leq 6$ censuses of Section 3, is the escape class that reappears for the zero-padding chain at every $d \geq 7$ (Section 4).

3 The spectrum at small lengths

The basin of a fixed point can range from nothing to everything. We first fix a scale for this range.

- **L0 (collapse).** Every admissible orbit reaches 0.
- **L1–L2 (fixed, partial).** $K(F) = F$ but the basin is a strict fraction of the space: possibly tiny, possibly large, in the worst case erratically either.
- **L3 (degenerate universal).** F is universal but $\mathrm{sv}_F(K) < d$: the universality is that of a lower-rank map written in d coordinates.
- **L4 (full universal).** F is universal with $\mathrm{sv}_F(K) = d$.
- **L5 (transcendent).** F, or an explicitly described family, is universal (or universal off an escape class, the inputs whose orbits reach 0) at infinitely many lengths under a fixed lifting recipe. Transcendence is *orthogonal* to the L3/L4 rank distinction, and the two liftings sit on opposite sides of it: the proven transcendent object of Section 4, 60714, is degenerate ($\mathrm{sv}_F < d$) and universal off an escape class above its roots, while the conjecturally transcendent duplication family of Section 5 is full ($\mathrm{sv}_F = d$). In the two chains this paper studies, which side a fixed point falls on is what decides whether its transcendence comes with a proof.

Proposition 3.1 (collapse is inhabited). *(Computed.) At $d = 4$, the rule with $c = (0, 0, 9, -9)$, nine times the gap between the two smallest sorted digits, sends all* 615 *admissible multisets to* 0*. There are exactly* 84 *collapse rules among the* 552 *ordered pairs at $d = 4$, and* 6 *among the* 30 *ordered pairs at $d = 3$.*

At the other end of L0–L2, this paper's census found that no fixed point of any rule at $d = 4$ has a one-multiset basin (Computed): strict bareness does not occur at four digits. What occurs instead, abundantly, is the partial fixed point, and the duplication family of Section 5 will furnish the canonical example of how wildly partial basins can behave inside one algebraically uniform family.

Theorem 3.2 (census). *(Computed; exhaustive over all full-variable rules and all admissible multisets, near-repdigits excluded.) The universal full-variable fixed points at $d = 3, 4, 5, 6$ number* 0*,* 4*,* 33*,* 506*.*

1. *At $d = 3$ there are none. The universal fixed points over all thirty rules are* 45*,* 450*,* 495*, each universal for a sign-flip pair of rank-*2 *rules ($c = \pm(99, 0, -99)$ for* 495*, $\pm(9, 0, -9)$ for* 45*, $\pm(90, 0, -90)$ for* 450*); their effective ranks are $\mathrm{sv}_F = 2$,* 1*,* 1*. The classical three-digit constant is a degenerate universal.*
2. *At $d = 4$ the four are $1746, 2538, 5382, 6174$, each universal for exactly one sign-flip pair of rules:*

$$1746 : \pm(9, -900, 900, -9) \qquad 2538 : \pm(90, 999, -999, -90)$$

$$5382 : \pm(900, -9, 9, -900) \qquad 6174 : \pm(999, 90, -90, -999).$$

   They form two anagram clusters, $\{1, 4, 6, 7\}$ and $\{2, 3, 5, 8\}$, and all have digit sum $\equiv 0 \pmod 9$ (as Lemma 2.2(a) forces for any fixed point).
3. *At $d = 5$ the thirty-three are listed in Appendix A. Four are degenerate at their fixed point despite full-variable rules: exactly the four whose sorted form carries a zero digit, since under a full-variable rule a zero digit drops out of the effective rank:* 54 *(sorted form $(5, 4, 0, 0, 0)$, $\mathrm{sv}_F = 2$),* 3753 *($\mathrm{sv}_F = 4$), and the two arrangements of $\{0, 1, 4, 6, 7\}$, namely* 60417 *and* 60714*, both with sorted form $(7, 6, 4, 1, 0)$ and $\mathrm{sv}_F = 4$. The other twenty-nine have $\mathrm{sv}_F = 5$. Note that* 60714 *is one of the degenerate four. The integer this paper follows to every length is, already at its native length, rank-deficient: universal for $c = (9900, 9, 90, -9000, -999)$ (in arrangement form $71460 - 10746 = 60714$), but acting through four coordinates, not five. Its single zero is the reason, and Section 4 shows the same zero is the reason it transcends.*
4. *At $d = 6$ the five hundred six stratify by zero count as $205, 240, 53, 8$ (zero through three zeros) and by digit sum as $8, 156, 244, 96, 2$ across sums $9, 18, 27, 36, 45$.*

The classical map is not full-variable at odd lengths (its borrow chain zeroes the middle coefficient, $c = (99, 0, -99)$ at $d = 3$), so the classical failure at $d = 5$ is the failure of a rank-4 map asked a rank-5 question. The thirty-three show the question itself has answers.

Theorem 3.3 (cross-dimensional selection). *(Computed.) Of the thirty-three universal full-variable fixed points at $d = 5$, exactly one,* 60714*, is again a universal full-variable fixed point at $d = 6$ under some rule. The $\{0, 0, 1, 4, 6, 7\}$ multiset carries exactly four of the* 506 *universals at $d = 6$:* 60714*,* 146070*,* 170460*,* 607140*.*

This was verified directly for this paper by intersecting the full $d = 5$ and $d = 6$ censuses. So 60714 is not chosen for its digits; it is the unique survivor of an exhaustive test. The next section shows that the survivor extends to every length (universal at its roots, and above them universal off an escape class, the inputs whose orbits reach 0) and why.

4 Lifting by zeros: 60714 at every length

Theorem 4.1 (zero-padding transcendence). *(Proven, with a finite-state component as stated below.) There is an explicit family of full-variable rules, one at each $d \geq 5$, related by a coefficient-preserving lifting, under which* 60714 *is a fixed point at every $d \geq 5$ (Proven), universal at $d = 5$ and $d = 6$ (Computed, exhaustive), and universal on $A_d \setminus E_d$ at every $d \geq 7$, where A_d is the admissible set and E_d, the escape class of Definition 4.6, is the set of inputs whose orbits reach* 0*. The reaching-time bound underlying the induction is proven algebraically for odd $d \geq 15$ and even $d \geq 18$ (Appendix E) and closed by complete enumeration for $7 \leq d \leq 16$ (about* 5.3 *million multisets, re-run in full for this paper); at the roots $d = 5, 6$, every non-repdigit multiset, near-repdigits included, reaches* 60714 *(Computed, Appendix E.4).*

The construction and the proof skeleton follow; every step uses the zeros.

The ladders. The native rule has $c^{(5)} = (9900, 9, 90, -9000, -999)$ and acts on sorted form $(7, 6, 4, 1, 0)$. For $d \geq 7$ odd, define $c^{(d)}$ by appending to $c^{(d-2)}$ the zero-sum pair $(+9 \cdot 10^{d-2}, -9 \cdot 10^{d-2})$ at the two new bottom ranks. The even ladder lifts the same way from the root

$$c^{(6)} = (9900,\ 9,\ 90,\ -9000,\ 99000,\ -99999),$$

the native -999 split across the two zero positions, $99000 - 99999 = -999$. Both ladders preserve the four coefficients at 60714's nonzero digits verbatim.

Proposition 4.2 (fixedness is free). *(Proven.) Every rule on either ladder fixes* 60714*.*

Proof. The sorted form of the padded fixed point is $(7, 6, 4, 1, 0, \ldots, 0)$. All appended (and split) coefficients sit at ranks holding zero digits, so $\sum_i c_i f_i = 9900 \cdot 7 + 9 \cdot 6 + 90 \cdot 4 - 9000 \cdot 1 = 60714$ at every length. ■

This was verified mechanically at every $d \leq 20$ and at $d = 100$ (Computed). The content of Theorem 4.1 is never the fixed-point equation; it is the basin, and the basin argument has three parts.

Lemma 4.3 (absorbing set). *(Proven.) Let T_d be the multisets of d digits, admissible or not (orbits pass through near-repdigits), whose sorted form ends in two zeros. For $d \geq 7$ on either ladder and $n \in T_d$,*

$$K^{(d)}(n) = K^{(d-2)}\big((s_0, \ldots, s_{d-3})\big), \quad \text{and} \quad K^{(d)}(n) \in T_d.$$

Proof. The appended pair multiplies $s_{d-2} = s_{d-1} = 0$ and vanishes; what remains is the rule two rungs down on the first $d-2$ sorted digits, since those coefficients are copied verbatim. The output is below 10^{d-2}, so its d-digit padding has at least two zeros, which sort to the tail. ■

Lemma 4.4 (core positivity). *(Proven.) For every sorted-descending s, the native part* $\mathrm{core}(s) = 9900s_0 + 9s_1 + 90s_2 - 9000s_3 - 999s_4$ *satisfies* $0 \leq \mathrm{core}(s) \leq 89{,}991 < 10^5$*.*

Proof. The upper bound drops the negative terms. For the lower bound: if $s_3 = s_4 = 0$ all terms are nonnegative. If $s_3 \geq 1, s_4 = 0$, then $\mathrm{core} = 9000(s_0 - s_3) + 900s_0 + 9s_1 + 90s_2 \geq 0$. If $s_3, s_4 \geq 1$, use $s_1 \geq s_4$ and $s_2 \geq s_3$, then $s_4 \leq s_3 \leq s_0$:

$$\mathrm{core}(s) \ \geq\ 9900s_0 + 9s_4 + 90s_3 - 9000s_3 - 999s_4 \ =\ 9900s_0 - 8910s_3 - 990s_4 \ \geq\ 9900(s_0 - s_3) \ \geq\ 0.$$

This completes the proof. ■

The even ladder's six-coefficient root obeys the same bounds one decimal place higher: expanding $99000s_4 - 99999s_5 = -999s_4 + 99999(s_4 - s_5)$ gives $\mathrm{core}_6(s) = \mathrm{core}(s_0, \ldots, s_4) + 99999\,(s_4 - s_5)$, and since $0 \le s_4 - s_5 \le 9$, $0 \le \mathrm{core}_6(s) < 10^6$. (The additive estimate $89{,}991 + 9 \cdot 99{,}999 = 989{,}982$ over-counts, its two extremes being incompatible; the exact maximum, computed in Appendix E.2, is 899,991.)

Lemma 4.5 (zeros are produced). *(Proven.) Each appended pair contributes $9 \cdot 10^{e_k}\,\delta_k$ with $\delta_k \ge 0$ a digit gap and the place e_k two higher than the previous pair's; the δ_k are disjoint adjacent differences in a sorted string, so $\sum_k \delta_k \le 9$. By Lemma 4.4 and its even-ladder extension the absolute value is never active and the output decomposes, without carries, into two-digit blocks (the digits of $9\delta_k$) sitting above a five-digit core (odd ladder) or six-digit core (even ladder). A block contributes two zero digits if $\delta_k = 0$, one if $\delta_k = 1$, none if $\delta_k \ge 2$; since $2\,\#\{\delta_k \ge 2\} + \#\{\delta_k = 1\} \le \sum_k \delta_k \le 9$, the output has at least $2M - 9$ zeros, M the number of pairs. For $M \ge 6$ (that is, $d \ge 17$ on the odd ladder, $d \ge 18$ on the even), this is at least two zeros: every orbit enters T_d in one step. A refined case analysis lowers the odd-ladder threshold to $d \ge 15$; it and the even ladder's full argument are in Appendix E.*

For $7 \le d \le 16$ the same conclusion (entry into T_d within at most eight steps) holds by complete enumeration over all admissible multisets, every length re-run in full for this paper; the table is in Appendix E. One detail of the induction needs a separate lemma. When Lemma 4.3 drops an orbit to length $d - 2$, the projected multiset can fail admissibility there: a repdigit projection is block-aligned (Definition 4.6) and belongs to the escape class, while a near-repdigit projection recovers admissibility within two further steps (Appendix E.4). Induction along each ladder then proves Theorem 4.1: orbits fall into T_d, where Lemma 4.3 reduces the length by two, the projection lemma hands the reduced orbit to the inductive hypothesis, and at the roots $d = 5, 6$ every non-repdigit multiset reaches 60714 (the census together with the near-repdigit check of Appendix E.4). Since 0 is block-aligned, E_d is the set of multisets whose orbits reach 0, and Lemma 4.3 carries the zero multiset at $d - 2$ to the zero multiset at d, so an input whose reduced orbit lies in E_{d-2} lies in E_d. The ladder rules themselves are written out through $d = 20$, with the $d = 100$ instance, in Appendix F.

Definition 4.6 (escape class). *The multisets constant on the native block and on each appended pair are called* block-aligned; *the escape class E_d is the backward orbit of the set of block-aligned multisets.* Block-aligned multisets are sent to 0 in one step, because the native coefficients sum to zero and each pair sees equal digits; since 0 is itself block-aligned, E_d is exactly the set of multisets whose orbits reach 0. At $d = 5, 6$ it contains no admissible multiset (the censuses show universality there, with no escape). At $d = 7$ the one-step part is the 45 multisets (x^5, y^2), $x > y$, and the full classification run for this paper found $|E_7 \cap A_7| = 81$ of 11,340 admissible multisets collapsing to 0 with all other 11,259 reaching 60714; at $d = 8$, $|E_8 \cap A_8| = 137$ of 24,210 (Computed). The one-step part is exactly the block-aligned set, at every d: conversely, if $K(n) = 0$, the carry-free decomposition of Lemma 4.5 into a nonnegative core and nonnegative blocks $9 \cdot 10^{e_k}\delta_k$ forces every $\delta_k = 0$ and core $= 0$; by the proof of Lemma 4.4 the core is positive when $s_4 = 0$ unless $s_0 = 0$, is at least $9900(s_0 - s_3)$ when $s_3, s_4 \ge 1$, and equals $999(v - s_4)$ when $s_0 = s_3 = v$, so it vanishes exactly when $s_0 = \cdots = s_4$; on the even ladder $\mathrm{core}_6 = \mathrm{core} + 99999(s_4 - s_5)$ vanishes exactly when $s_0 = \cdots = s_5$. The one-step part therefore has a closed form: the admissible block-aligned multisets number $\binom{9+k}{k} - 10$, k the number of blocks, which is 45 at both $d = 7$ and $d = 8$, as a direct count of the inputs with $K(n) = 0$ at each confirms (Computed). The classical map shows the same collapse at its own length (the multiples of 1111 fall to 0 in one step), though there the collapsing inputs are exactly the repdigits, which every convention already excludes; the escape class is that

same collapse, now reaching admissible inputs. The class is small at short lengths but grows overall with d: an independent recomputation in November 2026 found the share of non-repdigit multisets collapsing to 0 to be 0.56%, 3.12%, 6.66%, 9.72%, 12.06% at $d = 8, 12, 16, 20, 24$ (Computed; these shares take all non-repdigit multisets as denominator, not A_d). It is a weakening relative to the roots, stated in the theorem rather than hidden; a companion fixed point without this weakening is described in Section 8.

The contrast: 6174 does not climb. The classical constant admits the same kind of coefficient-preserving lifting: keep $(999, 90, -90, -999)$ on the nonzero digits of $(7, 6, 4, 1, 0, \dots)$, append zero-sum pairs. It never regains universality either. It acquires an escape class of its own (45 multisets at $d = 8, 9$ [S], fewer than the 137 of 60714 at $d = 8$), but no absorbing set proves its behavior at every length.

Theorem 4.7. *(Computed.) Padded* 6174 *has no full-variable fixed-point rule at all at* $d = 5$ *(none of the* 5,280 *rules fixes it, an algebraic obstruction, verified exhaustively here). At* $d = 6$ *exactly four full-variable rules fix it and the best basin is* 96.86% *of the* 4,905 *admissible multisets (verified exhaustively here). Under the lifting family at* $d = 7, 8, 9$ *the best basins are* 98.97%, 99.81%, 99.91% *[S], with the non-reaching inputs at* $d = 8, 9$ *being exactly the* 45 *multisets* (X^4, Y^4), $X > Y \geq 0$, *each collapsing to* 0 *[S].*

60714's native rule puts a nonzero coefficient, -999, on its zero digit, so the zero positions, where lifting must do its work, already carry weight in the rule. This is also why 60714 is degenerate at its native length (Theorem 3.2): the effective rank does not count weight resting on a zero. The absorbing set of Lemma 4.3 then reduces each length to the one two below. 6174 has no zero digit at its native length, so its lifts pad with zeros *outside* the native coefficients; no absorbing set reduces one length to the next, and nothing is known about its lifts beyond the lengths computed. The difference is one of proof, not of the size of the escape class. Where a fixed point keeps its zeros relative to its coefficients decides whether its behavior across lengths can be proven. That is the lesson of the zero-padding chain; the duplication chain that follows has no zeros at all.

5 Lifting by duplication: folding

Let $M_m = \{1, 4, 6, 7\}^m$, a multiset of length $d = 4m$ with sorted form $\mathbf{s}_m = (7^m, 6^m, 4^m, 1^m)$. The duplication question asks for universal fixed points with this digit content at every m. The right algebraic tool is an operation on rules.

Definition 5.1 (folding). *Let* $c = (c_0, \dots, c_{d_0-1})$ *be the coefficient vector of a rule at length* d_0 *and let* $k \geq 1$. *The* k*-fold* $c^{\otimes k}$ *at length* kd_0 *assigns to rank* $ki + j$ *(*$0 \leq i < d_0$, $0 \leq j < k$*) the coefficient* $c_i \cdot 10^{\,d_0(k-1-j)}$.

Proposition 5.2. *(Proven.) (a)* $c^{\otimes k}$ *is a valid rule (realizable by a permutation pair), full-variable if* c *is. (b) For every multiset* X *at length* d_0, *writing* X^k *for the multiset with each multiplicity multiplied by* k,

$$T^{\otimes k}(X^k) = \big(T(X)\big)^k :$$

the k*-fold states are closed under the folded rule, and the folded dynamics on them is conjugate to the base dynamics. (c) If* $K(W) = W$ *then the* k*-fold rule fixes the concatenation* $W^{(k)} = W \cdot \sum_{j<k} 10^{jd_0}$, *and every* k*-fold state* X^k *with* X *a non-repdigit in the base basin of* W *reaches* $W^{(k)}$.

Proof. (a) If $c_i = 10^{a_i} - 10^{b_i}$ with $\{a_i\}$ and $\{b_i\}$ each a complete residue list $\{0, \ldots, d_0 - 1\}$, then the folded coefficients are $10^{d_0(k-1-j)+a_i} - 10^{d_0(k-1-j)+b_i}$, and $\{d_0 t + a_i\}$, $\{d_0 t + b_i\}$ each enumerate $\{0, \ldots, kd_0 - 1\}$ exactly once. (b) The sorted form of X^k repeats each sorted digit of X at k consecutive ranks, so ranks $ki, \ldots, ki + k - 1$ all hold s_i, and

$$\sum_{i,j} c_i 10^{d_0(k-1-j)} s_i = R_k \cdot \sum_i c_i s_i, \qquad R_k = \textstyle\sum_{j<k} 10^{jd_0}.$$

Taking absolute values, $K^{\otimes k}(X^k) = R_k \cdot K(X)$, and since $K(X) < 10^{d_0}$ (it is a difference of two d_0-digit numbers), multiplication by R_k concatenates k copies of its padded digit string without carries. The digit multiset of the output is therefore $(T(X))^k$. (c) Immediate from (b). ■

Folding explains the duplication chain's classical landmarks directly.

Corollary 5.3. *(Proven.) The m-fold of the classical rule (the* interleaved *rule, with coefficients $999 \cdot 10^{4(m-1-j)}, 90 \cdot 10^{4(m-1-j)}, -90 \cdot 10^{4(m-1-j)}, -999 \cdot 10^{4(m-1-j)}$ on the four rank-blocks) fixes $F_m = 6174 \cdot R_m$, the string* 6174 *written m times, at every $m \geq 1$. Its restriction to m-fold states is conjugate to the classical four-digit map, so all of them lie in F_m's basin.*

So fixed points with the duplicated digit content exist at every m, provably. What the algebra does not control is everything off the folded subspace, and there the basins vary widely (Computed 5.4).

Computed 5.4 (the interleaved basins). *Over all non-repdigit multisets (the duplication-chain convention), the basin of F_m under the interleaved rule is*

m	d	states	basin of F_m
1	4	705	100%
2	8	24,300	100%
3	12	293,920	14.86%
4	16	2,042,965	99.69%
5	20	10,014,995	9.52%

*All five rows were enumerated in full for this paper. At $m = 3$ the rule has exactly two fixed points (*617461746174 *and* 535549955994*, the latter with a basin of* 100 *multisets), and* 250,148 *of the* 293,920 *states end in cycles.*

The sequence $100, 100, 14.9, 99.7, 9.5$ shows the pattern: one algebraically uniform construction, with basins scattered across the unit interval and no trend. The second fixed point at $m = 3$ also matters: it shows coefficient-positivity conditions (the rule is monotone in the sense below) do not force fixed-point uniqueness, a hypothesis the project's earlier notes briefly entertained and later refuted. It is also the first sighting of the structure that recurs at every failure of the duplication chain. The competitor 535549955994 is a relative of 549945, the classical six-digit Kaprekar constant $(995544 - 445599)$, built from the digits 4, 5, 9, and the same content shadows the chain elsewhere: the period-two cycle that breaks the classical rule's 2-fold of its 2-fold (Computed 6.2) is heavy in 4, 5, and 9, and the incoherent fixed multiset $\{6, 5^2, 4^5, 0^4\}$ that breaks the $k = 3$ fold (Computed 6.7) is heavy in 4 and 5. The shadow is a family rather than a single number: its members surface as fixed points at some multiplicities and as cycles at others, and they do not stabilize across m. This lack of stabilization blocks an induction that would exclude them (Section 7).

For the rest of the section, restrict to the rules adapted to M_m. Call a rule *pair-symmetric* if there are bijections between its seven-rank and one-rank coefficient blocks, and between its six-rank and

four-rank blocks, under which corresponding coefficients are negatives. Write $S_7 = \sum_{7\text{-block}} c_i$ and $S_6 = \sum_{6\text{-block}} c_i$, and let $C_j = c_0 + \cdots + c_j$ be the partial sums, so that for any sorted input, using $\sum c_i = 0$,

$$\sum_i c_i s_i \;=\; \sum_{j=0}^{d-2} C_j\, g_j, \qquad g_j = s_j - s_{j+1} \geq 0.$$

Proposition 5.5. *(Proven.) For a pair-symmetric rule at $d = 4m$: (a) $C_{m-1} = C_{3m-1} = S_7$ and $C_{2m-1} = S_7 + S_6$; (b) the value on the sorted form $\mathbf{s}_m$, whose only nonzero gaps are $g_{m-1} = 1$, $g_{2m-1} = 2$, $g_{3m-1} = 3$, is*

$$V \;=\; 6S_7 + 2S_6 \;=\; 4C_{m-1} + 2C_{2m-1};$$

(c) the rule fixes an arrangement of M_m if and only if $|V|$ has digit multiset M_m, a condition depending only on the partition of positions into the four blocks, not on the within-block coefficient assignment; (d) the interleaved rule is monotone ($C_j \geq 0$ for all j) at every m.

Proof. (a) Pair-symmetry gives $S_1 = -S_7$ and $S_4 = -S_6$, so $C_{3m-1} = S_7 + S_6 + S_4 = S_7 = C_{m-1}$. (b) Substituting into the gap form, $V = C_{m-1} + 2C_{2m-1} + 3C_{3m-1} = S_7 + 2(S_7 + S_6) + 3S_7 = 6S_7 + 2S_6$; directly, $V = 7S_7 + 6S_6 + 4S_4 + 1 \cdot S_1 = 6S_7 + 2S_6$. (c) The multiset map sends M_m to the digit multiset of $|V|$; if that is M_m the orbit of M_m is fixed and the fixed integer is $|V|$. Both S_7 and S_6 are sums of $10^a - 10^b$ over the block's position pairings, hence depend only on the position sets. (d) For the interleaved rule the partial sums climb through the seven- and six-blocks; inside the four-block, $C = 999R_m + 90R_m - 90\sum_{k\geq m-t} 10^{4k} \geq 999R_m \geq 0$ after t steps; inside the one-block, $C = 999(R_m - \sum_{k\geq m-t} 10^{4k}) \geq 0$. ■

Part (c) reduces the *fixing* question to explicit arithmetic over partitions (at $m = 2$, for instance, exactly 36 of the 2,520 partitions fix an arrangement of M_2; Computed), and part (d), with Corollary 5.3, says the algebraic requirements of the duplication conjecture (a fixing, monotone, full-variable rule at every m) are met uniformly. By Lemma 2.4, everything open lives in two dynamical conditions: no second fixed point, no cycle.

Computed 5.6 (the universal landscape, $m \leq 7$). *Universal fixed points of M_m exist at $m = 1, \ldots, 6$, and at $m = 7$ [N]. The counts of universal arrangements are* 2, 465*, and, from the project log [S],* 46, ≥ 313, ≥ 1, ≥ 2*. Within the pair-symmetric class at $m = 2$, this paper's exhaustive enumeration found* 312 *universal rules concentrated on* 12 *arrangements (Appendix C). The clean block point $F_m =$* 6174 *repeated is not universal under the interleaved rule for any $m \geq 3$ (its basins are the table above), and the $m = 3$ universals found are scrambled [S]; but block decomposition is not itself the obstruction, since at $m = 4$ and $m = 5$ folding makes block-decomposed arrangements universal,* 6174 *and* 1746 *each repeated m times among them (Computed 6.4). The following full-basin verifications were performed for this paper, each over the complete non-repdigit multiset space:*

m	d	states	verified universal	rule
3	12	293,920	666141417774	the $m = 3$ rule (re-verified)
4	16	2,042,965	1746 written four times	$(9, -900, 900, -9)^{\otimes 4}$
5	20	10,014,995	1746 written five times	$(9, -900, 900, -9)^{\otimes 5}$
6	24	38,567,090	666141417774 written twice	the $m = 3$ rule, 2-folded

*The $d = 20$ row is, to the author's knowledge, the first complete-enumeration universality verification at $m = 5$; the project's previous $m = 5$ witness (*14617461774617461746*) rested on a*

necessary-condition proxy [S]. The $d = 24$ row gives a second verified universal arrangement at $m = 6$ alongside the project's 666174141466617777741414 *[S].*

At $m = 7$ ($d = 28$) the clean repeat (6174 written seven times) is a universal fixed point under an explicit monotone pair-symmetric rule (repository file `multiplicity_chain/m7_witness.json`): all 124,403,610 non-repdigit multisets at $d = 28$ reach it [N]. This complete enumeration replaces an earlier check over all 4,491 multisets on $\{1, 4, 6, 7\}$ and a 500,000-input random sample. Like $m = 3$, it is reached by no tested fold (the 1746 rule's 7-fold fails, Computed 6.3); the naive fold leaks into a 4995/5355 cycle that the right within-block reordering removes.

Conjecture 5.7. *M_m carries a universal full-variable fixed point at every $m \geq 1$.*

The evidence now includes seven consecutive multiplicities with full-basin witnesses ($m = 7$ by complete enumeration [N]), and witness-count estimates that grow like $3 \cdot 10^3$, $2 \cdot 10^7$, $3 \cdot 10^8$ at $m = 3, 4, 5$ [S, sampling estimates calibrated against the exact $m = 3$ enumeration]. What it does not include is any uniform description: the universal arrangements for $m \geq 3$ are scrambled, and the structural features that characterize them at one multiplicity fail at the next when tested (Section 7). The table's most interesting column is the last one, and it is the subject of the next section: four of the six witnesses are folds.

6 When folding preserves universality

Proposition 5.2 transports the fixed point along $m \mapsto km$ for free; it says nothing about the basin off the folded subspace. The experiments reported here measure that. All "universal" verdicts below are complete enumerations; all "fails" verdicts are definitive, because the witness of failure is an admissible input (a multiset over $\{1, 4, 6, 7\}$) that demonstrably does not reach the fixed point. The *alphabet test* runs a rule on every multiset over $\{1, 4, 6, 7\}$ of the given length, and an *alphabet witness* is such a multiset that does not reach the fixed point. For a four-digit rule c, the level-j *tower element* is its j-times iterated 2-fold, at length $d = 4 \cdot 2^j$. The rules used are written out in Appendix D.

Computed 6.1 (first doubling: everything survives at $d = 4$). *The 2-folds of all eight universal full-variable rules at $d = 4$ (Theorem 3.2; four sign pairs $\pm c$, so four distinct maps) are universal at $d = 8$, each verified over all* 24,300 *multisets. In particular the 2-fold of the classical rule is the interleaved $m = 2$ rule, whose universality is the $m = 2$ row of Computed 5.4. This is a property of the four-digit bases, not a law: the 2-fold of* 60714*'s native five-digit rule fixes* 6071460714 *at $d = 10$ with basin* 53.34% *of the* 92,368 *non-repdigit multisets. The chain that lifts by zero-padding does not lift by doubling.*

Computed 6.2 (second doubling: selection). *Iterating the doubling, exactly one of the four $d = 4$ fixed points survives: the 2-fold of the 2-fold of $(9, -900, 900, -9)$, the* 1746 *rule, is universal at $d = 16$ (*2,042,965 *multisets), while the level-two tower elements of the* 2538*,* 5382*, and* 6174 *rules all fail, each trapping admissible $\{1, 4, 6, 7\}$-multisets. For the classical rule the obstruction is a period-two cycle through the multisets of* 5355517553558172 *and* 4995599449954176*, which captures* 14 *of the* 965 *alphabet multisets.*

Computed 6.3 (the 1746 fold spectrum). *For the* 1746 *rule $c = (9, -900, 900, -9)$ and $2 \leq k \leq 8$:*

k	d	verdict	basis
2	8	**universal**	complete enumeration, 24,300
3	12	fails (15.48%)	complete enumeration
4	16	**universal**	complete enumeration, 2,042,965
5	20	**universal**	complete enumeration, 10,014,995
6	24	fails	alphabet witness (523/2,921 reach F)
7	28	fails	alphabet witness
8	32	fails	alphabet witness (6,533/6,541)

The $k = 4$ entry is the direct 4-fold; the level-two tower element (a different within-block assignment on the same partition) is also universal at $d = 16$ (Computed 6.2). The 5-folds of the other three $d = 4$ rules fail; the 3-folds of all four fail. Mixed towers also fail where tested (3-fold of the 2-fold at $d = 24$: fails). The level-three tower element of the 1746 rule at $d = 32$, which differs from the direct 8-fold only in its within-block coefficient assignment, fixes 1746 written eight times and is universal over all 350,343,555 non-repdigit multisets [N]. This complete enumeration replaces an earlier check over all 6,541 alphabet multisets and 200,000 random admissible multisets.

Computed 6.4 (folding from $m = 2$: survival is common). *Of the 312 universal pair-symmetric rules at $m = 2$ (counting c and $-c$ separately; 156 distinct maps), exactly 144 (72 maps, spanning nine of the twelve universal arrangements) have universal 2-folds at $m = 4$, every one verified by complete enumeration over all 2,042,965 multisets. Among them are sixteen rules whose folds fix $F_4 = 6174$ written four times: the block fixed point that the interleaved rule fails to make universal (99.69%, Computed 5.4) is made universal by other doublings of the same partition, which explains, in folding terms, the project's observation that F_4 is universal only under scrambled within-block orderings [S]. The alphabet test was a perfect filter here: 144 of 312 doubled rules passed it, and all 144 proved universal.*

Computed 6.5 (folding from $m = 3$). *The 2-fold of the project's verified $m = 3$ universal rule is universal at $m = 6$: all 38,567,090 non-repdigit multisets at $d = 24$ reach* 666141417774666141417774.

Three things follow from this body of computation.

First, the duplication chain *does* have a working lifting, in instances. The project's earlier searches had tested and rejected a position-insertion lifting $m \to m + 1$ (best basin 3.07% across all natural variants [S]) and a family of m-periodic constructions (each universal at some multiplicities and provably cyclic at others [S]); the conclusion drawn was that universality does not transport. Folding shows the conclusion was too broad. The map $m \to km$ transports the entire algebraic package by Proposition 5.2, and the dynamics follows it in verified cases substantial enough to populate Computed 5.6's table: a single four-digit rule covers $m \in \{1, 2, 4, 5\}$ and a single twelve-digit rule covers $\{3, 6\}$.

Second, the survival pattern is irregular: $k = 2, 4, 5$ work from the 1746 root and $k = 3, 6, 7, 8$ do not, the direct 8-fold fails while the level-three tower element is universal [N], and the within-block assignment (the only difference between those two) is decisive. This is the duplication chain's recurring signature, now visible inside a single rule's fold family.

Third, doubling-survival is not rare: eight of eight rules from $m = 1$ (four of four distinct maps), 144 of 312 from $m = 2$, one of one tested from $m = 3$. At the only two doubling steps where the

survival fraction could be measured exhaustively it is 100% and 46%, which makes the truth of Conjecture 5.7 look like the generic outcome of an abundant mechanism rather than a sequence of coincidences. This is the strongest structural evidence the conjecture currently has. What *is* selective is iterated survival along a fixed tower: of the four $d = 4$ roots, only the 1746 rule's tower passes the second doubling, and it is universal at the third as well [N]. The selection is parallel to 60714's (one survivor of an exhaustive cross-dimensional test), and it nominates a single candidate for the duplication chain's L5 occupant.

Conjecture 6.6 (dyadic tower). *The iterated* 2*-fold of* $(9, -900, 900, -9)$ *is universal at* $d = 2^{j+2}$ *for every* $j \geq 0$*; equivalently,* 1746 *written* 2^j *times is a full universal fixed point at every dyadic multiplicity.*

This is the sharpest available subconjecture of Conjecture 5.7: a single explicit self-similar family, fixedness proven at every level, universality verified by complete enumeration at $j \leq 3$ ($j = 3$: [N]). A proof may be more accessible here than for the general conjecture. The earlier no-go evidence [S] rules out *m-periodic* uniform constructions (rules whose position structure repeats with the block) by exhibiting cycles at specific multiplicities. The folded towers are not periodic; their coefficient structure is self-similar across scales, with the base rule's geometry reproduced at every dyadic level, and the no-go arguments do not apply to them. What is missing is any analogue of Lemma 4.3: a closed set that the folded dynamics provably enters in bounded time and on which the multiplicity provably drops. The k-fold states are closed and reduce km to m (Proposition 5.2(b)), but nothing forces an orbit *into* them: they are closed but not absorbing. Finding a fold-adapted absorbing structure, playing the role zeros play in Section 4, is in the author's view the most concrete open *proof* problem this subject now offers.

That problem now has a candidate answer, found by asking what the fold does to states *off* the folded subspace. At length kd_0, rank $ki + j$ carries the coefficient $c_i\, 10^{\,d_0(k-1-j)}$, so for any sorted state x, folded or not,

$$K^{\otimes k}(x) \;=\; \Big|\sum_{j<k} 10^{\,d_0(k-1-j)}\; V\big(x^{(j)}\big)\Big|, \qquad x^{(j)} = (x_{ki+j})_{i<d_0},$$

where $V(t) = \sum_i c_i t_i$ is the base linear form and the *slices* $x^{(j)}$ are themselves sorted, the j-th entry of each rank-block. For the 1746 rule, $|V| \leq 8019 < 10^4$ on sorted tuples, so the slice values occupy disjoint decimal blocks of the output. Call x *coherent* when its nonzero slice values share a sign. On a coherent state the absolute value distributes across blocks: the image integer is the slice values written blockwise, its digit multiset is the disjoint union of theirs, and the entire future of the orbit therefore depends only on the tuple $(|V(x^{(0)})|, \ldots, |V(x^{(k-1)})|)$. The folded states are the coherent states with all slices equal; coherence enlarges this set to one that orbits are found to enter (Computed 6.7).

Computed 6.7 (coherence discriminates survival). *Every non-repdigit state becomes coherent within* 3 *steps at* $k = 2$ *(all* 24,300 *multisets) and within* 5 *steps at* $k = 4$ *(all* 2,042,965*); a* 300,000*-multiset random sample at* $k = 5$*,* $d = 20$ *enters within* 4 *steps with no exceptions. At* $k = 3$*, the failing fold, coherence is not reached from everywhere: the multiset* $\{9^8, 5, 4^3\}$ *falls into the multiset* $\{6, 5^2, 4^5, 0^4\}$*, a second fixed multiset of the* 3*-fold rule (the fixed integer is* 5,400,446,445*), incoherent with slice signs* $(+, +, -)$*. The fold's failure mode is an incoherent fixed point.*

Computed 6.8 (the pair quotient funnels at both tower levels). *At a tower level with slices* $x_{\text{even}}, x_{\text{odd}}$ *and level-below form* V*, let* P *be the set of pairs* $(|V(x_{\text{even}})|, |V(x_{\text{odd}})|)$ *over all coherent non-repdigit states* x*. On coherent states the factorization through slice values is exact:*

zero violations over the full enumerations. At tower level one ($k = 2$, $d = 8$), $|P| = 1{,}464$; the 1,202 whose forward orbits never exit coherence form a funnel: unique fixed pair $(1746, 1746)$, no cycles. At level two ($d = 16$) the tower element $(c^{\otimes 2})^{\otimes 2}$ is itself universal (a complete enumeration over all 2,042,965 multisets, new here alongside Computed 6.2); its states become coherent for the level-two pairing within 3 steps (against 4 for the flat four-slice decomposition of the same rule, and 5 for the direct 4-fold of Computed 6.7: the nested pairing is the dynamically natural one), and its pair set P, with 235,212 pairs, funnels to the unique fixed pair (17461746, 17461746) with no cycles.

The recursion between levels is syntactic. Writing V_j for the level-j form ($d_j = 4 \cdot 2^j$), the level above evaluates the level below on the two parity slices:

$$V_{j+1}(t) \;=\; 10^{\,d_j}\, V_j(t_{\text{even}}) + V_j(t_{\text{odd}}),$$

and the recursion is strong enough to decide signs outright.

Lemma 6.9 (sign and plateau). *(Proven.) (a) For every $j \geq 0$ and sorted x of length d_j: $V_j(x) < 0$ if and only if $x_{2^j} > x_{2^{j+1}}$; $V_j(x) = 0$ if and only if x is constant; $V_j(x) > 0$ otherwise. Moreover* $\max |V_j| \leq 10^{d_j} - 1$. *(b) Consequently a state x at level $j + 1$, with $a = 2^{j+1}$, $b = 2^{j+2}$, is incoherent (slice values of strictly opposite sign) exactly when it carries a critical* plateau: *either $x_a = x_{a+1} = \cdots = x_b > x_{b+1}$ with the even slice non-constant, or $x_a > x_{a+1} = \cdots = x_{b+1}$ with the odd slice non-constant.*

Proof. (a) Induction on j. At $j = 0$, $V_0 = 9(x_0 - x_3) - 900(x_1 - x_2)$: if $x_1 > x_2$ then $V_0 \leq 81 - 900 < 0$; if $x_1 = x_2$ then $V_0 = 9(x_0 - x_3) \geq 0$, vanishing exactly on constants; and $\max |V_0| = 8019 \leq 10^4 - 1$. For the step, $\max |V_{j+1}| \leq \max |V_j|\,(10^{d_j} + 1) \leq (10^{d_j} - 1)(10^{d_j} + 1) = 10^{d_{j+1}} - 1$; and since the inductive hypothesis already gives $\max |V_j| \leq 10^{d_j} - 1$, whenever $V_j(x_{\text{even}}) \neq 0$ the leading term $10^{d_j} V_j(x_{\text{even}})$ strictly exceeds $|V_j(x_{\text{odd}})|$ in magnitude and V_{j+1} takes its sign. The even slice $y_k = x_{2k}$ has $y_{2^j} = x_{2^{j+1}}$ and $y_{2^{j+1}} = x_{2^{j+2}}$, so the inductive rule transfers verbatim. If instead $V_j(y) = 0$, then y is constant, sortedness squeezes every odd rank between equal even neighbors ($x_0 = x_1 = \cdots = x_{d_{j+1}-2}$), and $V_{j+1} = V_j(x_{\text{odd}})$ with odd slice $(x_0, \ldots, x_0, x_{d_{j+1}-1})$, whose critical pair is interior and equal: nonnegative, zero exactly on constants. (b) Apply (a) to each slice: the even slice's critical pair is (x_a, x_b), the odd's is (x_{a+1}, x_{b+1}), and sortedness collapses the interval between equal ranks. ■

Mechanical confirmation found no violations over all sorted states at levels $0, 1, 2$ (715; 24,310; 2,042,975) and 120,000 adversarial states at level 3 (Computed). The sign of a 2^{j+2}-digit folded form is a comparison of two digits, and incoherence is a single shape: a run of $2^{j+1} + 1$ equal digits occupying ranks 2^{j+1} to 2^{j+2} or ranks $2^{j+1} + 1$ to $2^{j+2} + 1$ (roughly the second quarter of the sorted string), with a strict drop at one prescribed end. That shape is what bounded entry must destroy; at levels one and two it is destroyed with a proof rather than a sweep.

Proposition 6.10 (coherence entry, bound three). *(Proven for $j = 1, 2$; the finite checks over the gap families are computations.) At tower levels one and two, every non-repdigit state becomes coherent within three steps, meeting no repdigit on the way. The bound is sharp at both levels.*

Proof. By Lemma 6.9(b) an incoherent state is a critical plateau, and on a plateau the slice values collapse to closed forms in the digit gaps alone. At level one, type A has $V_0(\text{even}) = 9P$ with $P = x_0 - x_6 \geq 1$ (the equal critical pair kills the 900-term), and $V_0(\text{odd}) = 9R - 900Q$ with $Q = x_3 - x_5 \geq 1$ and $R = x_1 - x_7 \in [Q, 9]$, so the image value is $9P \cdot 10^4 - 9(100Q - R)$, a function of three gaps. Type B mirrors it: with $Q' = x_2 - x_4 \geq 1$, $P' = x_0 - x_6 \in [Q', 9]$ and $R' = x_1 - x_7 \geq 1$, the image value is $9(100Q' - P') \cdot 10^4 - 9R'$. The images form an 810-member

family (405 of each type) indexed by gaps, not by states; direct computation shows every member coherent or becoming so within two further steps (744 immediately, 62 in one, 4 in two). At level two ($a = 4$, $b = 8$) the recursion gives $V_1(y) = 10^4 V_0(y_0, y_2, y_4, y_6) + V_0(y_1, y_3, y_5, y_7)$, so the even slice's value reads ranks $(0, 4, 8, 12)$ and $(2, 6, 10, 14)$ and the odd slice's reads $(1, 5, 9, 13)$ and $(3, 7, 11, 15)$. Type A ($x_4 = \cdots = x_8 > x_9$) forces $x_4 = x_6 = x_8 = x_5 = x_7$; with the seven gaps $A_1 = x_0 - x_{12}$, $A_2 = x_2 - x_{14}$, $A_3 = x_8 - x_{10}$, $B_1 = x_1 - x_{13}$, $B_2 = x_8 - x_9$, $B_3 = x_3 - x_{15}$, $B_4 = x_8 - x_{11}$, the slice values and the image are

$$V_1(x_{\text{even}}) = 9 \cdot 10^4 A_1 + 9A_2 - 900A_3,$$

$$V_1(x_{\text{odd}}) = 10^4 \,(9B_1 - 900B_2) + 9B_3 - 900B_4,$$

$$K = 10^8 \, V_1(x_{\text{even}}) + V_1(x_{\text{odd}}),$$

with $1 \le B_2 \le A_3 \le B_4$ and $A_1, A_2, B_1, B_3 \in [B_4, 9]$ (sortedness: $x_9 \ge x_{10} \ge x_{11}$, and each of $x_0, \ldots, x_3$ is at least x_8 while each of $x_{12}, \ldots, x_{15}$ is at most x_{11}). Type B ($x_4 > x_5 = \cdots = x_9$) has, with $C_1 = x_0 - x_{12}$, $C_2 = x_4 - x_5$, $C_3 = x_2 - x_{14}$, $C_4 = x_5 - x_{10}$, $D_1 = x_1 - x_{13}$, $D_2 = x_3 - x_{15}$, $D_3 = x_5 - x_{11}$,

$$V_1(x_{\text{even}}) = 10^4 \,(9C_1 - 900C_2) + 9C_3 - 900C_4,$$

$$V_1(x_{\text{odd}}) = 9 \cdot 10^4 D_1 + 9D_2 - 900D_3,$$

$$K = -10^8 \, V_1(x_{\text{even}}) - V_1(x_{\text{odd}}),$$

with $C_2 \ge 1$, $0 \le C_4 \le D_3$, $C_2 + D_3 \le 9$, and $C_1, C_3, D_1, D_2 \in [C_2 + D_3, 9]$. These closed forms and constraints follow by hand from Lemma 6.9 and the recursion, by substituting the plateau equalities and using sortedness; the non-constancy conditions of Lemma 6.9(b) hold automatically, and V_2 takes the sign of the even slice's value, positive for type A and negative for type B, which fixes the sign in each formula for K. As a check, they reproduce the slice values and image of all 251,940 incoherent states of the full census. The abstract family, one image per gap tuple satisfying the constraints, has 63,855 members of each type, 127,710 in all; direct computation shows every member coherent or becoming so within two further steps (123,919 immediately, 3,646 in one, 145 in two). No orbit of a non-repdigit state meets a repdigit, at any level j. By Lemma 6.9(a) a non-repdigit state has $V_j \neq 0$, so its image is a positive multiple of 9 (Lemma 2.2), and the induction of Lemma 6.9(a) gives $\max |V_j| \le 8019\,(10^{d_j} - 1)/9999 < 10^{d_j} - 1$. A d-digit repdigit $r \cdots r$ has digit sum rd, and $d_j = 4 \cdot 2^j$ is prime to 9, so a repdigit that is a multiple of 9 has $r = 9$, that is, it equals $10^{d_j} - 1$, which exceeds every image. Hence the image of every non-repdigit state is a non-repdigit. Sharpness: the full enumerations attain entry time three at both levels. ■

(Evidence at $j = 3$: 60,000 incoherent $d = 32$ states constructed directly from the plateau shape all enter coherence within three steps, the same bound. The level-three enumeration [N] implies entry within some bound at $j = 3$, since every orbit reaches the coherent fixed point, but not the bound three. We conjecture entry within three steps at every level.)

What the quotient is for is excluding cycles, and one failure mode needs more care than Computed 6.8 gives it. The funnel there is verified on the pairs whose orbits never exit coherence; but nothing yet excludes a cycle that weaves through coherent and incoherent states alternately: its coherent arcs leave the partial pair system before returning, so a funnel over never-exiting pairs cannot see it, and bounded entry alone does not kill it. Hence the two properties verified so far do not by themselves imply the conjecture. The repair is to quotient by the *return map*. A coherent state's image multiset is the disjoint union of its blocks' digits (a function of the pair alone), so the entire future of a coherent state is determined by its pair. Let $\widehat{T}$ send a pair to the pair of the first

coherent state on the forward orbit of its assembled image; we say $\widehat{T}$ is total on the pair set P of Computed 6.8 when it is defined at every pair of P and takes its values in P, and bounded entry makes it defined there.

Theorem 6.11 (return-map reduction). *(Proven.) Fix a tower level with folded fixed point F, the level below's fixed value W written twice. Suppose* (E′)*: every non-repdigit state becomes coherent within a bounded number of steps, meeting no repdigit en route; and* ($\widehat{\text{F}}$)*: $\widehat{T}$ is total on P with no repdigit met inside any excursion, (W, W) is its unique fixed pair, and it has no cycle of period ≥ 2. Then the tower element is universal.*

Proof. The state space is finite, so an admissible orbit ends in a periodic set C. The orbit stays off repdigits and 0 forever: before first coherence by (E′), and excursion by excursion afterwards by ($\widehat{\text{F}}$), noting that an excursion could also *end* at a repdigit, which is vacuously coherent with pair $(0, 0)$; but $(0, 0) \notin P$ (a coherent state with both slice values 0 has constant slices, and sortedness then makes it a repdigit), so this is excluded by $\widehat{T}$ taking values in P. So C consists of non-repdigit states, and by (E′) it contains a coherent state s^* with pair p^*. The coherent states met in order around the cycle form a $\widehat{T}$-orbit returning to p^*: a $\widehat{T}$-cycle, which by ($\widehat{\text{F}}$) can only be the fixed pair (W, W). The state following s^* is then the assembled fixed point F, so $F \in C$, and F being fixed forces $C = \{F\}$. Every admissible orbit ends at F. ■

Computed 6.12 (the return-map funnel, complete at both levels). *At level one, $\widehat{T}$ is total on P (all* 1,464 *pairs, the* 262 *whose orbits exit coherence included), with no repdigit excursions, unique fixed pair* $(1746, 1746)$*, and no cycles. At level two the same holds on P (all* 235,212 *pairs), with unique fixed pair* $(17461746, 17461746)$*. Together with Proposition 6.10 this discharges both hypotheses of Theorem 6.11 at $j = 1, 2$: a second, quotient-route derivation of universality at both levels, independent of the basin enumerations that first established it.*

The classical four-digit map reduces to a 54-state gap system funneling to $(6, 2)$ (Section 7); the coherent fold dynamics reduces, one level up, to the 1,464-pair return map funneling to $(1746, 1746)$; the level above, to 235,212 pairs funneling to the fixed pair doubled. The boundary between pairs that keep coherence and pairs that lose it (which no tested inequality or congruence describes) no longer needs describing: the return map absorbs the excursions, and the difficulty measured in Section 7 now sits in the funnel property of $\widehat{T}$ itself. Theorem 6.11 reduces Conjecture 6.6 to two uniformities: bounded coherence entry at every level (proven at $j = 1, 2$ with sharp bound three; implied with some bound at $j = 3$ by the complete enumeration [N], and with bound three in every level-three test), and the funnel property of every level's return map (complete at $j = 1, 2$, and implied at $j = 3$ by the same enumeration, since a second fixed pair or a cycle of $\widehat{T}$ on P would lift to an orbit avoiding F). The repdigit clauses of both hypotheses hold automatically at every level of this tower, by the argument at the end of the proof of Proposition 6.10, so what remains is coherence entry and the funnel. Both are open for $j \geq 4$, and the obstacle has been measured. The entry arguments are finite checks in the digit gaps, and the one reduction that would keep such a certificate bounded across levels (dependence on the leading gap-windows only) fails when measured: image coherence at level two depends on the low-order gaps too (921 of 1,710 coarse classes mix outcomes), so the family this style of certificate must check grows without bound in the level. A uniform entry lemma needs a genuine carry-and-block argument, one level above Lemma 4.5. Section 7 states the sharpened problem.

A remark on what folding cannot do even in principle. A fold realizes multiplicity m only from a universal base at some m_0 properly dividing m. From $m_0 = 1$ every m is in reach, but a fold reaches m only when it survives, which Computed 6.3 shows is not for free: the 1746 rule's k-folds fail at

$k = 3, 6, 7, 8$. Primes such as $m = 7, 11, 13$ therefore need either a surviving fold of a four-digit rule or a witness of their own. Folding reorganizes the conjecture around its prime levels; it does not exhaust it.

7 The cost of a certificate

Every universality statement in this paper is, by Lemma 2.4, the statement that a finite functional graph has one root and no cycles. Such statements always have proofs (trace the graph), and the question that separates Section 4 from Section 6 is whether they have proofs *smaller than the graph.* For the zero-padding chain they do: Lemmas 4.3–4.5 are a certificate of total size independent of d. This section measures the same question at the base of the duplication chain and finds the answer sharply different.

The classical four-digit map factors through two coordinates. For sorted digits $a \geq b \geq c \geq e$, the map computes $999(a - e) + 90(b - c)$, so with $p = a - e \in \{1, \ldots, 9\}$ and $q = b - c \in \{0, \ldots, p\}$ the dynamics lives on 54 states, with unique fixed point $(6, 2)$ (this is 6174: $p = 7 - 1$, $q = 6 - 4$). The system is acyclic with maximal reaching time 6, and its iterated images shrink as $54, 20, 14, 10, 7, 4, 1$ (Computed). A *polynomial Lyapunov function of degree D* is a $\Phi \in \mathbb{R}[p, q]$, $\deg \Phi \leq D$, with $\Phi(\text{next}) < \Phi(\text{current})$ at all 53 non-fixed states; its existence is a linear program in the $\binom{D+2}{2}$ coefficients.

Theorem 7.1 (the Lyapunov degree of Kaprekar's theorem is exactly seven). *(Proven.) No polynomial Lyapunov function of degree ≤ 6 exists for the four-digit gap system: for each $D = 2, \ldots, 6$ there is an exact rational Farkas certificate (a nonnegative rational combination of the 53 decrease constraints summing to a contradiction), verified in integer arithmetic (the $D = 4$ certificate is supported on 15 states; Appendix B). A Lyapunov function of degree 7 exists: an explicit rational polynomial, verified in exact arithmetic to decrease strictly along all 53 transitions.*

It does not say the convergence is unprovable: it is provable by a 54-state enumeration, and indeed by a degree-7 polynomial, drawn from a 36-dimensional space. It also does not say (as a first pass over this project's data once suggested) that certificates exist only at the trivial interpolation threshold, degree 9, where polynomials span all functions on the state set and the reaching time itself becomes a vacuous witness; degree 7 is strictly below that threshold, so a non-tautological algebraic certificate exists. What the theorem says is quantitative: the cheapest polynomial witness of the seventy-seven-year-old fact needs degree 7, which lives in the 36-dimensional space of polynomials of degree at most 7, on a 54-point system. The certificate is found by search, not derived from structure, and there is no visible way to write it as the first member of a family indexed by m.

That is the correct statement of the base-case obstruction, and it propagates. Each of the standard strategies for Conjecture 5.7 was pursued in this project and failed for a reason now identifiable with the base case's lack of transportable structure; the table records the sharp form of each failure (all [S], from the project's verified computational log; the basin and enumeration figures quoted were re-verified here where marked).

strategy	sharp obstruction
uniform / periodic construction	the one clean recipe universal at $m = 3, 4$ fails at $m = 2$ and $m = 5$ (period-8 and period-16 cycles); position-insertion lifting $m \to m{+}1$: best basin 3.07%

strategy	sharp obstruction
induction on excluded cycles	obstruction cycles do not stabilize: the $m = 5$ cycle families do not occur at $m = 3, 4$; each multiplicity grows new ones
structural characterization of universal rules	the property characterizing universal partitions at $m = 3, 4$ (seven-block at the top) yields zero universals at $m = 5$ in an 11.3-million-sample search; the verified $m = 5$ winner has a different shape
Lyapunov / monovariant	LP-infeasible over digit-count and quadratic feature classes at $d = 16$; at the base, Theorem 7.1
modular invariant	no modulus in $2, \ldots, 300$ separates the basin from the cycles; mod 9 (Lemma 2.2) is universal and hence empty
union bound / local lemma over cycles	the dominant cycle alone captures $\sim$67% of candidate rules, far too dense
counting lower bound	the universal fraction is erratic (6.94% exact at $m = 3$, $\sim$28% at $m = 4$, $\sim$0.04% at $m = 5$); only the absolute count grows, and no provable lower bound for it is known

The common root is uniformity: universal rules exist in abundance at every tested m, but the *description* of where they sit changes with m, which simultaneously defeats construction, induction, characterization, and any local-to-global probabilistic argument. A workable proof must either find structure that is scale-free (the dyadic towers of Section 6 are the first candidate family of that kind to survive testing) or dispense with description entirely and prove existence by a counting argument robust to wild fluctuation in the universal fraction. Both are research programs, not techniques.

Compare Section 4. The zero-padding proof never confronts any of this, because it never proves convergence at a fixed length at all: it proves a *reduction between lengths* (Lemma 4.3) and pays a bounded, d-independent cost to reach the reducing set (Lemma 4.5). The duplication chain now has both pieces, in partly proven form: the reduction is a theorem valid at every level (Theorem 6.11), and the reaching is proven at the first two levels with sharp bound three (Proposition 6.10). What blocks the ascent has been measured: the reaching proof is a finite gap-certificate whose size grows with the level, and the funnel half remains a computation. The precise frontier, sharpened by Section 6's findings:

Open Problem 7.2. *For the dyadic tower of the* 1746 *rule, prove uniformly in the level j the two hypotheses of Theorem 6.11, both established at $j = 1, 2$ and implied at $j = 3$ by the complete enumeration [N]:* (E') *every non-repdigit state becomes coherent within a bounded number of steps, meeting no repdigit en route, proven there with sharp bound three (Proposition 6.10), and conjectured to hold with bound three at every level; and* ($\widehat{\mathrm{F}}$) *the return map $\widehat{T}$ funnels on P: unique fixed pair, no cycles (Computed 6.12). The repdigit clauses of both hold automatically at every level (proof of Proposition 6.10). By Theorem 6.11 the two together imply Conjecture 6.6. For* (E')*, Lemma 6.9 reduces the task to showing that the image of a critical plateau cannot sustain critical plateaus for more than two further steps; the gap-certificate style of argument used at $j \leq 2$ does not climb, by the measurement in Section 6, and what is needed is a carry-and-block argument one level above Lemma 4.5. For* ($\widehat{\mathrm{F}}$)*, no enumeration can reach every level (the level-three result [N] is a basin enumeration, not a pair-system analysis), and the funnel boundary admits no simple description; any proof must exploit the syntactic recursion $V_{j+1} = 10^{d_j} V_j \oplus V_j$.*

Open Problem 7.3. *Prove any lower bound on the number of universal rules for M_m that is positive for infinitely many m.*

8 Further directions (November 2026)

This paper follows one integer across lengths. Since it was written the question has been turned around: choose a fixed point first, and ask for a rule that sends every input to it, with a proof that is not an enumeration. That inverse problem is open. This section records, without proofs, what has been found on the way. The labels of Section 1 do not apply here, with one exception: the two complete enumerations named under Independent recomputation are cited in the body with the tag [N] (Section 9). Where a result below is called proven, its proof is written out in the project repository. It has been checked by independent machine passes, by AI review, and by complete enumeration where stated. No human referee has checked it, and it is not part of this paper.

Independent recomputation. The censuses of Theorem 3.2 (0, 4, 33, 506) and the selection of Theorem 3.3 were recomputed twice with new code, in November 2026, and agree. At $d = 5$ the rule `adcbe` $-$ `deacb` sends every one of the 99,990 non-repdigit five-digit strings, near-repdigits included, to 60714 in at most 30 steps. Two further results came out of the same runs. First, if rules that leave a letter in place are allowed, and only repdigits are excluded, the five-digit universal fixed points number 54 and not 33 (74 if near-repdigits are also excluded), and 22 of the 54 recur at $d = 6$; the uniqueness of 60714 in Theorem 3.3 is a statement about full-variable rules at both lengths. Second, two items earlier supported only by sample-based checks are now complete enumerations, and the text above cites them with the tag [N]: the dyadic tower at level three (all 350,343,555 non-repdigit multisets at $d = 32$) and the clean repeat at $m = 7$ (all 124,403,610 at $d = 28$).

Why the sorting step. Dropping the sort (taking the digits as written and subtracting one fixed shuffle of the positions from another) was tested completely at $d = 4$ and $d = 5$ (276 and 7,140 pairs of shuffles, every start). No such pair is universal. The typical one sends about 98% of five-digit starts into cycles; the best sends 99,840 of 100,000 starts to a single nonzero integer (some rules send every start to 0).

Fixed points at every length in every base. For every length $L \geq 6$ and every base $b \geq 3$ the repository gives an explicit rule, a pair of arrangements in which the second is the first with its two halves exchanged (at six digits, `cadbfe` $-$ `bfecad`; one middle letter stays fixed when L is odd), under which every non-repdigit reaches $(b-1)(b^h-1)$, $h = \lceil L/2 \rceil$ (proven in the repository; not refereed). For rules of this type in even bases, universality is equivalent to a check on a small table of states (126 at ten digits in base 10), and a rule passing a slightly stronger test lifts to every longer even length (both proven in the repository for even bases only; not refereed). The fixed points reached this way all have one shape, a block of digits followed by its digit complement, and the seed rules are found by search.

Fixed points built to order. In base 10, among integers that are a difference of two arrangements of their own digits (call them targets), the share for which a search finds a universal rule grows broadly with length: 2 of 54 at six digits, 142 of 165 at ten, 146 of 148 at thirteen (by search; no construction, and no proof that the remaining targets have no rule).

A caution about bases. Rules that are universal in every base from 3 to 36 can fail at base 50 and beyond. Two kinds of failure have been observed: cycles whose digits grow in proportion to the base, and cycles near the largest and smallest digits that look the same in every large base. Agreement across small bases is not evidence for all bases.

Open. The inverse problem; a construction for the seeds of the lifting statement above; odd bases; a proof of the base-free description of the second kind of failure; and human review of everything in this section.

8.1 A companion: 146070 with no escape class

At six digits the rule is `bfadce` − `cbdeaf`, with coefficient vector

$$c^{(6)} = (990,\ 90000,\ -99990,\ -900,\ -99,\ 9999);$$

on $(7, 6, 4, 1, 0, 0)$ it computes $607140 - 461070 = 146070$. Longer rules extend this core: the letter `f` of the first string is replaced by the last letter, and the letters from `f` on are written in front in reverse order, offset by one (`f` through the second-to-last letter before the first string, `g` through the last letter before the second). At $d = 8$ the rule is `gfbhadce` − `hgcbdeaf`. In coefficients, for $d \geq 7$,

$$c^{(d)} = (990,\ 90000,\ -99990,\ -900,\ -99,\ 999999,\ 9 \cdot 10^6,\ \ldots,\ 9 \cdot 10^{d-2},\ 10^4 - 10^{d-1}),$$

the entry $9 \cdot 10^j$ sitting on s_j for $6 \leq j \leq d - 2$.

In the November 2026 audit, complete enumeration at every length from 6 to 24 found that every non-repdigit multiset, near-repdigits included, reaches 146070 (38,567,090 multisets at $d = 24$), in at most 42 steps to the integer at every length tested: no cycles, no other fixed point, no escape to 0. An independent re-run with new code in November 2026 confirmed this at every length through 30 (211,915,122 multisets at $d = 30$) and reproduced the closure counts. A computer-assisted argument for every $d \geq 6$ is written out in the project repository. It has been reviewed by AI only, has not been refereed, and is not part of this paper's Proven claims.

It works because the digit the rule writes at each place $k \geq 6$ is the gap $s_{k-1} - s_k$ between neighboring sorted digits (up to one borrow), and the gaps sum to at most 9: after one step the image has, besides a run of 9s, at most fifteen nonzero digits, whatever d is. Further steps shorten a run of six or more 9s by five (once by four). Once a zero is present the map no longer depends on d, and a finite closure computation over 2,941,883 multisets of nonzero digits, every one reaching $\{7, 6, 4, 1\}$ without exceeding 20 nonzero digits, settles every $d \geq 21$; enumeration covers the rest. The bound it gives is $\lfloor (d - 1)/5 \rfloor + 43$ steps.

Theorem 4.1's rule family leaves an escape class at every $d \geq 7$, and the audit found no full-variable rule that makes 60714 itself universal at $d = 7$ or $d = 8$. 146070, the same digits with one more zero and already one of the four six-digit universals on $\{0, 0, 1, 4, 6, 7\}$ in Theorem 3.3, has no escape class at any length tested. At every length tested it is reached by every non-repdigit input, the property Theorem 4.1 lacks; the computer-assisted argument extends this to every $d \geq 6$. Nor is 146070 the only number of this kind. The same audit ran the closure test on all 190,800 full-variable six-letter rules extended by this recipe: 551 pass, and complete enumeration through $d = 20$ found each of them universal for a single fixed integer from some first length on, 75 of them from $d = 6$. Among the 551, 146070 is the only one with exactly four nonzero digits that are all different: $7, 6, 4, 1$, the digits of 6174. The only other family with four nonzero digits, $\{9, 9, 8, 1\}$, repeats one.

9 Methods and provenance

All computations are over digit multisets; a rule's orbit depends only on the sorted digit vector, which is what makes complete enumeration feasible: $\binom{d+9}{9}$ states minus exclusions, against 10^d integers. Conventions: the censuses of Section 3 and the 60714 chain exclude repdigits and near-repdigits (615, 1,902, 4,905, 11,340, 24,210 admissible multisets at $d = 4, \ldots, 8$); the duplication chain excludes repdigits only (24,300; 293,920; 2,042,965; 10,014,995; 38,567,090 at $d = 8, 12, 16, 20, 24$). Basins were computed by orbit tracing with terminal memoization; the $d = 20, 24$ enumerations use a combinatorial ranking of multisets into a flat byte array (at most $\binom{d+9}{9}$ bytes of state), which is what brings the previously "infeasible" $d = 20$ complete basin (Computed 5.6) to under two minutes in an interpreted language. The Lyapunov LPs of Theorem 7.1 were solved with an exterior solver and then *re-verified in exact rational arithmetic*: the infeasibility certificates are nonnegative rational vectors checked to annihilate the constraint matrix over $\mathbb{Q}$, and the degree-7 function was rationalized and its 53 strict decreases checked exactly.

Results marked [S] are quoted from the project's computational log and were not re-executed for this paper; they comprise the 6174 lifting basins at $d = 7, 8, 9$, the exhaustive one-step closure confirmations at $d = 17, 18, 20$ cited in Appendix E, the duplication-chain counts at $m \geq 3$ and their witness-count estimates, the $m = 3$ complete pair-symmetric enumeration (3,000 universal rules among 43,200 monotone fixing rules), the obstruction-cycle and no-go analyses summarized in Section 7's table, and the $m = 6$ witness 666174141466617777741414. Results marked [N] are complete enumerations carried out in November 2026 after the body of the paper was written; they are reported in Section 8, their scripts are in the repository, and they were not re-run for this paper's provenance statement. The tag marks exactly two results: the level-three tower element at $d = 32$ (350,343,555 multisets) and the $m = 7$ clean repeat at $d = 28$ (124,403,610 multisets). The $d \leq 16$ reaching-time enumeration behind Theorem 4.1 (about 5.3 million multisets) was re-run in full for this paper (Appendix E.3), as were the one-step escape counts at $d = 7, 8$ and the $d = 6$ census lists of Appendix A. Everything else was computed afresh for this paper from the engine: the full censuses at $d \leq 6$ with the collapse counts and the selection theorem, the 60714 ladder checks through $d = 100$, the near-repdigit projection closed forms confirmed through $d = 401$, the full classifications at $d = 7, 8$, the five interleaved basins, the two-fixed-point inventory at $d = 12$, the $m = 2$ pair-symmetric enumeration, every fold verdict in Section 6 (including the 144 complete $d = 16$ basin enumerations of Computed 6.4 and the complete enumerations at $d = 8, 20, 24$), the coherence-entry and quotient analyses of Computed 6.7–6.8 (complete enumerations at $d = 8, 12, 16$ including the tower element's universality, with the $k = 5$ entry bound from a 300,000-multiset random sample at $d = 20$), the sign-lemma sweeps behind Lemma 6.9 (complete at levels zero through two, 120,000 adversarial states at level three), the abstract gap families of Proposition 6.10 (810 members at level one and 127,710 at level two, with the level-two closed forms checked against all 251,940 incoherent states of the census), the return-map closures of Computed 6.12 (1,464 and 235,212 pairs), the level-three plateau sampling (60,000 constructed incoherent states plus 50,000 random at $d = 32$) together with the leading-window dependence probe (1,710 coarse classes), the full $d = 18$ one-step closure re-run of Appendix E.3, the direct near-repdigit checks at the ladder roots behind Appendix E.4, the letter-to-coefficient expansion of every row of Appendix F's table, and both directions of Theorem 7.1. The verification scripts accompany the manuscript.

Acknowledgments

The enumerations, searches, and verification harness were built and run in collaboration with AI models, Anthropic's Claude and OpenAI's ChatGPT, used as research instruments; several of the project's intermediate claims (a uniqueness hypothesis for monotone rules; a first reading of the Lyapunov threshold) were retracted or corrected when verification contradicted them, and the corrected statements appear above. The questions, the direction, and the responsibility for the results are the author's.

A The census lists

54, 3753, 12456, 14562, 15642, 16524, 16578, 16758, 17685, 18342, 21456,

21834, 24156, 24183, 24561, 28539, 37584, 37854, 38754, 41562, 41832, 42183,

43758, 43785, 43875, 45612, 45621, 53928, 58239, 60417, 60714, 65781, 67581.

All thirty-three were found by exhaustive enumeration of the 5,280 full-variable rules against the 1,902 admissible multisets. The four with $\mathrm{sv}_F < 5$ are exactly those whose sorted form contains a zero digit: 54 ($\mathrm{sv}_F = 2$), 3753 ($\mathrm{sv}_F = 4$), and the two arrangements 60417, 60714 of $\{0, 1, 4, 6, 7\}$ ($\mathrm{sv}_F = 4$ each). All four are degenerate universals (the five-digit analogues of the degenerate three-digit constants), and of the thirty-three only 60714 survives to $d = 6$ (Theorem 3.3).

At $d = 6$, the high-zero strata of the 506, where cross-dimensional behavior concentrates, are small enough to list, and both lists below were checked against this paper's recomputed census. The eight fixed points with three zero digits are all arrangements of the single multiset $\{0, 0, 0, 2, 2, 5\}$:

252, 2520, 20025, 25200, 200025, 200250, 250200, 252000.

The fifty-three with two zero digits fall into eight multiset clusters, of sizes 32 ($\{9, 9, 8, 1, 0, 0\}$), 7 ($\{5, 5, 4, 4, 0, 0\}$), 5 ($\{9, 7, 1, 1, 0, 0\}$), 4 ($\{7, 6, 4, 1, 0, 0\}$, the thread of Theorem 3.3: 60714, 146070, 170460, 607140), 2 ($\{7, 7, 3, 1, 0, 0\}$), and three singletons (9225, 67023, 52605). The full sorted list:

4545, 7191, 8919, 8991, 9189, 9225, 10899, 17019, 37017, 44505, 52605, 54450,

60714, 67023, 80919, 80991, 81099, 89019, 90819, 90918, 91809, 98091, 100899,

108099, 109809, 146070, 170460, 180909, 189009, 190089, 190809, 445005,

445050, 504450, 544500, 607140, 700191, 701901, 707130, 719100, 809091,

809109, 810909, 890091, 900891, 901890, 908091, 908109, 908190, 908901,

909081, 910089, 910809.

The remaining 445 fixed points (205 with no zero digit, 240 with one) are in the repository's census file.

B The degree-4 Farkas certificate

For $D = 4$ the Lyapunov LP has 15 unknowns and 53 constraints, one per non-fixed gap state (p, q), each of the form $w \cdot (\mathbf{m}(G(p,q)) - \mathbf{m}(p,q)) \leq -1$ with $\mathbf{m}$ the monomial vector of degree ≤ 4 and G the gap map. The exact infeasibility certificate found and verified for Theorem 7.1 is a rational vector $y \geq 0$, supported on the fifteen states

$$(1,0),\ (2,0),\ (4,2),\ (4,3),\ (5,1),\ (5,3),\ (5,5),\ (6,0),\ (6,3),\ (7,1),\ (7,3),\ (8,0),\ (9,2),\ (9,6),\ (9,9),$$

with common denominator 10,885,528,810 and numerators, in the order listed, 10,006,016, 65,632,092, 2,410,551,428, 371,776,120, 1,301,323,953, 435,495,967, 53,435,169, 10,685,478, 1,755,787,268, 783,903,408, 2,132,993,086, 1,219,059,176, 171,653,279, 154,623,519, 8,602,851 (the null space on this support is one-dimensional, so y is determined up to scale), satisfying $\sum_s y_s(\mathbf{m}(G(s)) - \mathbf{m}(s)) = 0$ exactly and $\sum_s y_s = 1$: any Φ of degree ≤ 4 would have to decrease along a nonnegative combination of transitions that exactly cancels, which is absurd. Certificates for $D = 2, 3, 5, 6$ have supports of sizes 6, 10, 21, 28 and were verified the same way. The degree-7 Lyapunov function has 36 rational coefficients and decreases strictly along all 53 transitions, its largest increment $\Phi(\text{next}) - \Phi(\text{current})$ being -0.979 after rationalization (the normalization ≤ -1 above is only a scaling convention, met after multiplying Φ by $1/0.979$); its coefficient vector is in the accompanying repository.

C The pair-symmetric universal rules at $m = 2$

Exhaustive enumeration for this paper: 2,520 ordered partitions of the eight positions into the four blocks, of which 36 fix an arrangement of M_2 (Proposition 5.5(c)); 64 within-block coefficient assignments each; 312 of the resulting rules are universal over all 24,300 non-repdigit multisets, fixing 12 distinct arrangements with multiplicities

$$14617746,\ 17461746,\ 17746146 \text{ (48 rules each)};\quad 61461774,\ 61746174,\ 61774614 \text{ (32)};$$

$$14177466,\ 17417466,\ 17741466 \text{ (16)};\quad 14661774,\ 17466174,\ 17746614 \text{ (8)}.$$

Twenty-four of the 312 are monotone. The fold experiments of Computed 6.4 take this set as their base.

D Rules used in Section 6

The twelve-digit universal rule (Computed 5.6, $m = 3$; re-verified here over all 293,920 multisets) has coefficient vector

$$c = (99999999999,\ 9990000000,\ 999900000,\ 99999000,\ 999900,\ 9990,$$

$$-99999000,\ -999900,\ -9990,\ -99999999999,\ -9990000000,\ -999900000)$$

and fixed point 666141417774; it is pair-symmetric and is not itself a fold. Its 2-fold (Definition 5.1, $d_0 = 12$, $k = 2$) is the $d = 24$ rule of Computed 6.5. The k-folds of the 1746 rule are fully determined by Definition 5.1 from $c = (9, -900, 900, -9)$: rank $5i + j$ of the 5-fold at $d = 20$ carries $c_i \cdot 10^{16-4j}$, giving

$$(9 \cdot 10^{16}, 9 \cdot 10^{12}, 9 \cdot 10^{8}, 9 \cdot 10^{4}, 9,\ -900 \cdot 10^{16}, \ldots, -900,\ 900 \cdot 10^{16}, \ldots, 900,\ -9 \cdot 10^{16}, \ldots, -9),$$

with fixed point 1746 written five times. The iterated dyadic tower at $d = 32$ is $\big((c^{\otimes 2})^{\otimes 2}\big)^{\otimes 2}$, which differs from $c^{\otimes 8}$ in the within-block assignment of equal-magnitude coefficients to ranks; the former is universal at $d = 32$ [N], the latter fails the alphabet test (Computed 6.3).

In the letter notation of Section 2 the folds are visible to the eye. Label the sorted positions $a, b, c, \ldots$; at $d = 4m$ the ranks fall into four blocks of m equal digits, and copy j of base rank i is the j-th letter of block i. The 1746 rule and two of its surviving folds ($k = 2$ and $k = 5$) read:

m	d	π	σ	$\pi \cdot s - \sigma \cdot s$
1	4	`cbad`	`bcda`	$4671 - 6417$
2	8	`ecag fdbh`	`cega dfhb`	$46714671 - 64176417$
5	20	`kfap lgbq mhcr nids ojet`	`fkpa glqb hmrc insd jote`	$4671\cdots - 6417\cdots$

(Spaces are cosmetic.) Each four-letter group is the base pattern `cbad` (respectively `bcda`) written in the letters of successive copies, so the two rearrangements evaluate to 4671 and 6417 repeated m times, and the subtraction is the base subtraction $4671 - 6417 = -1746$ carried out blockwise: the pattern $\texttt{cbad}\,\texttt{cbad}\cdots - \texttt{bcda}\,\texttt{bcda}\cdots$, in exact analogy with the classical $\texttt{abcd} - \texttt{dcba}$. Every row was machine-checked to fix 1746 written m times (Computed). The same notation for the 60714 zero-padding ladder appears in the final appendix; there the rungs grow by prepended letter-pairs instead of repeated blocks, which is the two liftings' difference made typographic.

E The reaching-time bound in full

This appendix proves the reaching-time half of Theorem 4.1 at full rigor: every admissible orbit enters the absorbing set T_d within a bounded number of steps, the bound being one step for odd $d \geq 15$ and even $d \geq 18$, and at most eight steps for $7 \leq d \leq 16$ by complete enumeration. It closes with the projection lemma that the induction of Section 4 quietly uses.

The decomposition. By Lemmas 4.4 and 4.5 the absolute value in $K^{(d)}$ is never active, and the output splits without carries into a core plus disjoint two-digit blocks:

$$K^{(d)}(x) \;=\; \mathrm{core}(x) + \sum_{k=1}^{M} 9 \cdot 10^{e_k}\, \delta_k, \qquad \delta_k = x_{e_k} - x_{e_k+1} \in \{0, \ldots, 9\},$$

with M pairs at places e_k ascending by two. Writing $9\delta_k = 10u_k + v_k$, the block at places (e_k, e_k+1) carries digits (v_k, u_k): two zeros if $\delta_k = 0$, one zero ($u_k = 0$) if $\delta_k = 1$, none if $\delta_k \geq 2$, since $9\delta_k \in \{18, 27, \ldots, 81\}$ then has both digits nonzero. Each block value is at most $81 < 100$, so blocks never carry into one another; the core is below 10^5 (odd) or 10^6 (even) by Lemma 4.4 and its even-ladder extension, so nothing carries out of it either.

E.1 (odd ladder, $d \geq 15$). *(Proven.) For odd $d \geq 15$, every multiset x of d digits, admissible or not, satisfies $K^{(d)}(x) \in T_d$.*

Proof. Let Z_0, Z_1, Z_{2+} count the pairs with $\delta_k = 0$, 1, ≥ 2. The blocks contribute $2Z_0 + Z_1$ zero digits, and from $2Z_{2+} + Z_1 \leq \sum_k \delta_k \leq 9$ (Lemma 4.5),

$$2Z_0 + Z_1 \;=\; 2M - Z_1 - 2Z_{2+} \;\geq\; 2M - 9.$$

For odd $d \geq 17$, $M = (d-5)/2 \geq 6$ and the block region alone has ≥ 3 zeros. At $d = 15$, $M = 5$ gives only ≥ 1, and the second zero comes from a case split on x_4:

Case $x_4 = 0$. Sorting forces $x_5 = \cdots = x_{14} = 0$, every $\delta_k = 0$, and the block region is ten zeros.

Case $1 \leq x_4 \leq 7$. The five δ_k are alternate adjacent differences among positions $5, \ldots, 14$, so their sum telescopes below $x_5 - x_{14} \leq x_5 \leq x_4 \leq 7$, whence $2Z_0 + Z_1 \geq 10 - 7 = 3$.

Case $x_4 \in \{8, 9\}$. Sorting forces $x_0, \ldots, x_3 \in \{8, 9\}$, and a direct check of the two subcases $x_0 = x_3$ and $x_0 = x_3 + 1$ gives $\mathrm{core}(x) \leq 999$ and $\mathrm{core}(x) \leq 9999$ respectively; in both, $\mathrm{core}(x) < 10^4$, so decimal place 4 of the output is zero, joining the ≥ 1 block zero. ■

E.2 (even ladder, $d \geq 18$). *(Proven.) For even $d \geq 18$, every multiset x of d digits, admissible or not, satisfies $K^{(d)}(x) \in T_d$.*

Proof. The even core core_6 occupies places 0–5 (its exact maximum over the 5,005 sorted 6-tuples is 899,991 $< 10^6$, attained at $(9, 9, 9, 9, 9, 0)$), so it never carries past place 5, and the appended pairs stack directly above it at places $(6, 7), (8, 9), \ldots$ (Appendix F). With $M' = (d-6)/2$ pairs the block count gives $2Z_0 + Z_1 \geq 2M' - 9$ as in E.1, which is ≥ 2 as soon as $M' \geq 6$, that is $d \geq 18$. ■

E.3 (the enumerated range). *(Computed; every row re-run in full for this paper.) For $7 \leq d \leq 16$, the maximum number of steps for an admissible orbit to enter T_d, over all admissible multisets on the appropriate ladder, is*

d	admissible multisets	max steps to T_d
7	11,340	8
8	24,210	6
9	48,520	3
10	92,278	3
11	167,860	2
12	293,830	2
13	497,320	2
14	817,090	2
15	1,307,404	1
16	2,042,875	1

The $d = 15, 16$ rows confirm one-step closure where E.1 applies and just below the even threshold; exhaustive one-step confirmations at $d = 17$ (3,124,450 multisets), $d = 18$ (4,686,725), and $d = 20$ (10,014,905) are in the project log [S], and the $d = 18$ confirmation (the even threshold itself) was re-run in full for this paper: all 4,686,725 admissible multisets enter T_{18} in one step, zero violations (Computed).

E.1–E.3 together give a uniform reaching-time bound at every $d \geq 7$, which is Lemma 4.5's role in the induction. One gap remains: when an orbit lands in T_d and Lemma 4.3 projects it to length $d - 2$, the projected multiset may not be admissible there.

E.4 (projection lemma). *(Proven; the root cases $d-2 \in \{5, 6\}$ Computed.) Let $n \in T_d$ lie on the orbit of an admissible input (n itself need not be admissible), with sorted form $(x_0, \ldots, x_{d-3}, 0, 0)$, and write m for the length-$(d-2)$ projection. Exactly three things can happen. If m is admissible, the inductive hypothesis applies. If m is a repdigit $(v, \ldots, v)$, then n is block-aligned, $K^{(d)}(n) = 0$,*

and n, hence the input, lies in E_d. If m is a near-repdigit $\{v^{d-3}, w\}$, for any of the ninety pairs with $v \in \{0, \ldots, 9\}$ and $w \neq v$, the orbit recovers an admissible projection within two further steps and the induction proceeds.

Proof of the recovery claim. The states of the near-repdigit case at d are the multisets $\{v^{d-3}, w, 0, 0\}$ with $v \in \{0, \ldots, 9\}$, $w \neq v$ (ninety (v, w) pairs; the nine with $v = 0$ are near-repdigits at d and arise only as images, such as 900000 in Section 2's example at $d = 7$), and project to the near-repdigit $\{v^{d-3}, w\}$ at length $d - 2$. On a near-repdigit exactly one adjacent gap of the sorted string is nonzero (at the top rank if $w > v$, at the bottom if $w < v$), so, writing the value in the gap form $\sum_j C_j g_j$ with C_j the coefficient partial sums, the image is a single partial-sum multiple of that gap, in closed form on either ladder.

If $w < v$, the nonzero gap sits at the bottom rank: the core partial sums all cancel and only the final appended pair fires, $K^{(d-2)} = 9(v-w) \cdot 10^{d-4}$. For $v - w \geq 2$ this is two distinct nonzero digits above $d-4$ zeros, admissible at $d-2$, so the orbit recovers in one step. For $v - w = 1$ it is $9 \cdot 10^{d-4}$, padding to the near-repdigit $(9, 0, \ldots, 0)$; one further step gives $K^{(d-2)} = 9 \cdot 9900 = 89{,}100$, padding to $(9, 8, 1, 0, \ldots, 0)$, admissible. Recovery in at most two steps.

If $w > v$, the nonzero gap is at the top rank and only the leading partial sum (the coefficient $c_0 = 9900$ itself) fires: $K^{(d-2)} = 9900\,(w - v)$ on both ladders. Equivalently, w carries c_0 while every v carries the rest, and $9 + 90 - 9000 + 99000 - 99999 = -9900$ collects the even root's v-coefficients to the identical value. This runs through $9900, 19{,}800, \ldots, 89{,}100$ as $w - v = 1, \ldots, 9$ (the value 9 only when $v = 0$), each admissible at $d - 2$ (top digits $9, 9$ when $w - v = 1$, otherwise three distinct nonzero digits) above $d - 4$ zeros. Recovery in one step.

Only the ladder roots $d - 2 \in \{5, 6\}$, lacking the appended-pair structure, fall outside the closed forms; there all ninety near-repdigits, any of which can arise as a projection, were checked directly to reach 60714 (as were all 1,992 and 4,995 non-repdigit multisets); the near-repdigits do so in at most 29 steps at $d - 2 = 5$ and 15 at $d - 2 = 6$ (29 and 18 over all non-repdigit multisets) (counted to the digit multiset of 60714; one further step can be needed to reach the integer itself, 30 at $d - 2 = 5$) (Computed; a separate check, since the censuses exclude near-repdigits and so cannot be cited for them). The closed forms hold at every length, so the induction is closed uniformly in d. Direct enumeration of the near-repdigit cases only (the eighty-one pairs with $v \neq 0$, not all multisets) confirms them (seventy-two pairs recovering in one step, the nine with $v - w = 1$ in two), together with total reaching times (to the digit multiset) at most 27 steps on the odd ladder and 15 on the even, for every $7 \leq d \leq 401$; the nine pairs with $v = 0$ fall under the $w > v$ closed form. ■

F The ladder, explicitly

Labeling sorted positions $a = s_0, b = s_1, c = s_2, \ldots$ in descending order, a rule is two letter-strings whose k-th letter names the position feeding decimal place 10^{d-1-k} of the respective rearrangement. The ladder of Theorem 4.1 reads:

d	ladder	π	σ
5	odd (root)	`adcbe`	`deacb`
6	even (root)	`eadcbf`	`fdeacb`
7	odd	`fgadcbe`	`gfdeacb`
8	even	`gheadcbf`	`hgfdeacb`

d	ladder	π	σ
9	odd	`hifgadcbe`	`ihgfdeacb`
10	even	`ijgheadcbf`	`jihgfdeacb`
11	odd	`jkhifgadcbe`	`kjihgfdeacb`
12	even	`klijgheadcbf`	`lkjihgfdeacb`
13	odd	`lmjkhifgadcbe`	`mlkjihgfdeacb`
14	even	`mnklijgheadcbf`	`nmlkjihgfdeacb`
15	odd	`nolmjkhifgadcbe`	`onmlkjihgfdeacb`
16	even	`opmnklijgheadcbf`	`ponmlkjihgfdeacb`
17	odd	`pqnolmjkhifgadcbe`	`qponmlkjihgfdeacb`
18	even	`qropmnklijgheadcbf`	`rqponmlkjihgfdeacb`
19	odd	`rspqnolmjkhifgadcbe`	`srqponmlkjihgfdeacb`
20	even	`stqropmnklijgheadcbf`	`tsrqponmlkjihgfdeacb`

Each rung extends the rung two above by one prepended letter-pair, encoding the appended zero-sum pair $(+9 \cdot 10^{d-2},\ -9 \cdot 10^{d-2})$ at the two new bottom ranks (the same signs on both ladders, the lower-rank letter leading in π and trailing in σ), and the four coefficients $(9900, 9, 90, -9000)$ at 60714's nonzero digits recur verbatim in every row. Every row was machine-checked to expand to the coefficient vector of Section 4's ladder (Computed). The order of each pair's letters matters: crossing them between π and σ on the even rungs flips every appended sign and gives a different map. That map still fixes 60714, since the appended coefficients sit on its zeros, but its escape class and reaching times differ from those of Section 4 ($|E_8 \cap A_8| = 45$ in place of 137). The fixed-point equation certifies nothing about a ladder beyond the fixed point. Under the fixed-point assignment $a = 7$, $b = 6$, $c = 4$, $d = 1$, all later letters zero, every row evaluates to $\pi \cdot s = 71460$ and $\sigma \cdot s = 10746$, so $K = |71460 - 10746| = 60714$ (machine-checked).

The recipe is mechanical at any length. At $d = 100$ (even ladder): positions 0–3 carry the locked coefficients $(9900, 9, 90, -9000)$, positions 4–5 the split pair $(99000, -99999)$, and positions 6–99 carry forty-seven appended zero-sum pairs $(+9 \cdot 10^{2k+6},\ -9 \cdot 10^{2k+6})$ at ranks $(2k + 6,\ 2k + 7)$. Every coefficient from position 4 on multiplies a zero digit of the padded fixed point, and

$$K(60714) \ = \ |\,9900 \cdot 7 + 9 \cdot 6 + 90 \cdot 4 - 9000 \cdot 1\,| \ = \ |69300 + 54 + 360 - 9000| \ = \ 60714,$$

the same four-term arithmetic at every length (machine-checked at $d = 100$). What is engineered here is only the fixed-point equation; the basin, which is the theorem, is Section 4 and Appendix E.

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